Math 211-2015S-W9-Fri (Pg 1)

Review The Fundamental Theorem of Calculus

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

Marginal Cost

$$\int_a^b C'(q)\,dq = C(b) - C(a)$$

The area under the marginal cost curve is the difference in total cost

Often rewritten

$$F(b) = F(a) + \int_a^b F'(t)\,dt$$

$$C(b) = C(a) + \int_a^b C'(t)\,dt$$

$$\underbrace{C(b)}_{\text{total cost}} = \underbrace{C(0)}_{\text{Fixed Cost}} + \underbrace{\int_0^b C'(t)\,dt}_{\text{total variable cost}}$$

New

Antiderivatives

If f is the derivative of F
i.e. if $F'(x) = f(x)$
Then F is an antiderivative of f

$$F \underset{\text{antiderivative.}}{\overset{\text{deriv}}{\rightleftarrows}} f$$

Eg: $x^2 \xrightarrow{D} 2x$ — Take derivative: $2x$ is the derivative of x^2

Then: $2x \xrightarrow{AD} x^2$ — Find an antiderivative: then x^2 is an antiderivative of $2x$

Now consider $x^2+1 \underset{AD}{\overset{D}{\rightleftarrows}} 2x$

Therefore x^2+1 is (another) antiderivative of $2x$

In fact, x^2+C, for any ~~[illegible]~~ constant C is an antiderivative of ~~the~~ $2x$.

In fact ~~[illegible]~~ all antiderivatives of $2x$ have the form x^2+C

We think of ~~[illegible]~~ "x^2+C" as a <u>family</u> of functions. Each member of the family is an antiderivative of $2x$ and together the form the family of <u>all</u> antiderivatives".

The family of antiderivatives of a function is called the indefinite integral of the function. It's always has form

$F+C$ where $F \underset{AD}{\overset{D}{\rightleftarrows}} f$

one member of the family

a unique function

~~[illegible]~~

Written $\int f(x)\,dx = F(x)+C$

note no limits of integration

Compare with definite integral $F \underset{AD}{\overset{D}{\rightleftarrows}} f$

limits of integration → $\int_a^b f(x)\,dx$ is a number $=F(b)-F(a)$ (area under graph if positive between a and b

no limits → $\int f(x)\,dx$ is a family of functions $=F(x)+C$

Finding Formulas:

An antiderivative of x is $\frac{x^2}{2}$

$\frac{d}{dx}\left(\frac{x^2}{2}\right) = x$ / $\int x\,dx = \frac{x^2}{2} + C$

An antiderivative of x^2 is $\frac{x^3}{3}$

$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$ / $\int x^2\,dx = \frac{x^3}{3} + C$

An Antiderivative of $x^n = \frac{1}{n+1} x^{n+1}$

$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ / $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

What if $n = -1$ this formula doesn't work

we'll deal with this later.

An antiderivative of k (constant) is kx

$\frac{d}{dx} kx = k$ $\quad\int k\,dx = kx + C$

Properties

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c f(x)\,dx = c \int f(x)\,dx$$

A $$\int (3x + x^2)\,dx = 3\int x\,dx + \int x^2\,dx$$

$$= 3\frac{x^2}{2} + \frac{x^3}{3} + C$$

can take derivative to check

$3x + x^2$ ✓

only need to add constant once

B $$\int (q^3 - 6q^2)\,dq$$

$$= \frac{q^4}{4} - \frac{6q^3}{3} + C = \frac{q^4}{4} - 2q^3 + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

When $n=-1$ this suggests: $\int x^{-1}dx = \underbrace{\frac{1}{0}x^0}_{\text{not defined}}$

$\int \frac{1}{x}dx$ has another formula (above formula doesn't work for $n=-1$)

What function has a derivative $\frac{1}{x}$?

$$\int \frac{1}{x}dx = \ln(x) + C$$

Because $\frac{d}{dx}\ln(x) = \frac{1}{x}$

Note $\frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot -1 = \frac{1}{x}$

But here's the thing

$\ln(x)$ only exists for $x>0$

$\ln(-x)$ only exists for $x<0$

For negative x, $\ln(-x)$ is ~~the~~ an anti derivative of $1/x$
For positive x, $\ln(x)$ is an antiderivative of $1/x$

For negative x $\quad |x| = -x$
For positive x $\quad |x| = x$

So for both negative and positive x

$\ln(|x|)$ is an antiderivative of
$\frac{1}{x}$ for all x (except 0)

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq 1 \\ \ln|x| + C & \text{if } n = 1 \\ & (x \neq 0) \end{cases}$$

$$\int e^x\,dx = e^x + C \qquad \left(\frac{d}{dx} e^x = e^x\right)$$

$$\int e^{kx}\,dx = \frac{1}{k} e^{kx} + C \qquad \left(\frac{d}{dx}\frac{1}{k} e^{kx} = e^{kx}\right)$$

$$\int \cos x\,dx = \sin x + C \qquad \left(\frac{d}{dx}\sin x = \cos x\right)$$

$$\int \sin x\,dx = -\cos x + C \qquad \left(\frac{d}{dx}(-\cos x) = \sin x\right)$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$$

$$\left(\frac{d}{dx}\frac{1}{k}\sin(kx) = \cos kx\right)$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$$

$$\frac{d}{dx}\left(-\frac{1}{k}\cos(kx)\right) = \sin(kx)$$

Examples

① $\int (\sin x + 3\cos(5x))\,dx$

$= -\cos x + \frac{3}{5}\sin(5x) + C$

② $\int (e^x + x^2 + 3\sin x)\,dx$

$= e^x + \frac{x^3}{3} + (-3)\cos x + C$

$\boxed{\text{Evaluating Definite Integrals}}$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

the constant C cancels here

F can be any antiderivative of f

eg $\int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = e - 1$

antiderivative

eg. $\int_3^4 (x^2 + \cos x)\,dx = \frac{x^3}{3} + \sin x\Big|_3^4 = \frac{4^3}{3} + \sin(4) - \frac{3^3}{3} - \sin(3)$